LINEAR ALGEBRA I - BACKPAPER EXAM.

Answer all questions. You may use results proved in class after correctly quoting them.

(1) (a) Find all solutions of the system of equations $AX = B$ where

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

[10]

(b) Let A, B be two $n \times n$ matrices. If $BA = I$ show that A is invertible, and B is its inverse. [10]

(c) Let A be a square matrix. Show that if $AX = B$ has a unique solution for some particular column vector B, then it has a unique solution for all B. [5]

(2) (a) Let A be a square matrix. Use row reduction to prove that $\det(A) = \det(A^t)$. [5]

(b) Let B be the matrix

$$B = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 \\ 1 & 2 & -2 & 1 \end{pmatrix}$$

Exhibit a basis for the row space and a basis of the column space of B. Justify. [10]

(c) A real matrix B is said to be skew-symmetric if $B = -B^t$. Show that if B is a $n \times n$ skew-symmetric matrix, then $(I - B)$ has rank n. [10]

(3) (a) Let V be the vector space of all real polynomials of degree at most 4. Show that every subspace W of V is the kernel of a linear transformation $T : V \longrightarrow \mathbb{R}^5$. Is every subspace W of V the kernel of a linear map $S : V \longrightarrow \mathbb{R}^4$? [10+5]

(b) Prove that $B = ((1,2,0)^t, (2,1,2)^t, (3,1,1)^t)$ is a basis of $\mathbb{R}^3$. Find the coordinate vector of the vector $v = (3,2,1)^t$ with respect to this basis. Let $B' = ((0,1,0)^t, (1,0,1)^t, (2,1,0)^t)$. Determine the basechange matrix from B to B'. [15]

(c) Determine the dimensions of the kernel and image of the linear operator T on the space $\mathbb{R}^n$ defined by $T(x_1, x_2, \ldots, x_n)^t = (x_1 + x_n, x_2 + x_{n-1}, \ldots, x_n + x_1)^t$. [10]

(4) What do you mean by a generalized inverse of a matrix? Find a generalized inverse G of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

and hence describe (using G) the complete set of solutions of the system [2+4+4]

$$AX = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$